Note Title 9/30/2019

Definition: A <u>pre-log structure</u> on a scheme X is a sheaf of monoids along a morphism

$$\alpha_X : (\mathcal{M}, +) \longrightarrow (\mathcal{O}_X, \cdot)$$

It's a <u>log structure</u> if $\alpha^{-1}(\mathcal{O}_X^*) \cong \mathcal{O}_X^*$

A <u>morphism</u> of log schemes consists of $f: X \to Y$ w/ a morphism $f^{-1}\mathcal{M}_Y \xrightarrow{f^\#} \mathcal{M}_X$

s.t

$$\begin{tikzcd} f^{-1}\mathcal{M}_Y \arrow[r, "f^\#"] \arrow[d, "\alpha_Y"'] & \mathcal{M}_X \arrow[d, "\alpha_X"] \\ f^{-1}\mathcal{O}_Y \arrow[r, "f^*"] & \mathcal{O}_X \end{tikzcd}$$

ex. Every scheme has a trivial log structure $\begin{array}{c}(X, \mathcal{O}_X^*) \\ \downarrow \\ \mathcal{O}_X\end{array}$

ex. $D \subseteq X$ divisor

$$\mathcal{M}_{(X,D)}(U) = \left\{ g \in \mathcal{O}_X(U) \;\middle|\; g|_{U \setminus D} \in \mathcal{O}_X(U \setminus D)^* \right\}$$

In particular, if $U \cap D = \phi$

then $\mathcal{M}_{(X,D)}(U) = \mathcal{O}_X^*(U)$

ex. $\mathbb{k}$: field $\rightsquigarrow$ $\mathcal{M}_{\mathbb{k}} = \mathbb{k}^* \oplus Q$ w/ $Q^* = \{0\}$

$$\begin{aligned} \alpha_X : \mathcal{M}_{\mathbb{k}} &\longrightarrow \mathbb{k} \\ (x,q) &\longmapsto \begin{cases} x, & \text{if } q=0 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

Two special choices of Q

① $Q = 0$ $\rightsquigarrow$ trivial log structure

② $Q = \mathbb{N}$ $\rightsquigarrow$ standard log point

ex. Given $\alpha : \underset{\sim}{P} \longrightarrow \mathcal{O}_X$ prelog structure

sheaf of monoids

there is a natural associated log structure

$$\begin{aligned} \mathcal{M}_X := P \oplus \mathcal{O}_X^* \big/ \{(p, \alpha(p)^{-1}) \mid p \in \alpha^{-1}(\mathcal{O}_X^*)\} &\longrightarrow \mathcal{O}_X \\ (p,f) &\longmapsto f\alpha(p) \end{aligned}$$

In particular, $\underline{f} : \underline{X} \longrightarrow \underline{Y}$ w/ log structure $(\mathcal{M}_Y, \alpha_Y)$ on Y

then $f^{-1}\mathcal{M}_Y \xrightarrow{\alpha_Y} f^{-1}\mathcal{O}_Y \xrightarrow{f^*} \mathcal{O}_X$

$\rightsquigarrow$ $f^*\mathcal{M}_Y$ log structure on $\underline{X}$

Definition: $(X, \mathcal{M}_X) \xrightarrow{f} (Y, \mathcal{M}_Y)$ is <u>strict</u> if $\mathcal{M}_X = f^*\mathcal{M}_Y$

There is an short exact sequence of monoids on a log scheme.

$$1 \to \mathcal{O}_X^* \xrightarrow{\alpha_X^{-1}} \mathcal{M}_X \longrightarrow \overline{\mathcal{M}}_X \longrightarrow$$

characteristic/ghost sheaf

Carries combinatorial information about log structures

ex. $X = \operatorname{Spec} k$ $\quad Y = \operatorname{Spec} k[x] = \mathbb{A}^1$

$$f : X \longrightarrow \mathbb{A}^1$$
$$\bullet \longmapsto 0$$

$$\mathcal{M}_{(\mathbb{A}^1,0)}(\mathbb{A}^1) = \{ kx^n \mid n \geq 0 \}.$$

$\Rightarrow f^* \mathcal{M}_{(\mathbb{A}^1,0)} =$ standard log structure on $\operatorname{Spec} k$.

$$f^{-1}\mathcal{M}_{(\mathbb{A}^1,0)} \oplus k^* / \sim \xrightarrow{\cong} \mathbb{N} \oplus k^*$$
$$(\phi x^n, s) \longmapsto (n, \phi(0) s)$$

Definition: P = monoid $\rightsquigarrow$ $\underline{P}$ constant sheaf

A chart for a log scheme $(X, \mathcal{M}_X)$ is a morphism

$\underline{P} \longrightarrow \mathcal{M}_X$ s.t.

$$\begin{array}{ccc} P & \longrightarrow & \mathcal{M}_X \\ & \searrow & \downarrow \\ & & \mathcal{O}_X \end{array}$$

the associated log structure is isomorphic to $\mathcal{M}_X$

Definition: A log structure $\mathcal{M}_X$ is <u>fine</u> if there is an étale open cover $\{U_i\}$ of X s.t on each U_i there is a finitely generated monoid P_i s.t P_i is a chart for $\mathcal{M}|_{U_i}$.

ex. $\sigma \subseteq M_{\mathbb{R}}$ strictly convex

$X_\sigma = \operatorname{Spec} k[\sigma^\vee \cap N]$ $\quad P = \sigma^\vee \cap N$

$$\begin{array}{ccc} P & \longrightarrow & k[P] \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ n & \longmapsto & x^n \end{array} \quad \approx \quad \underline{P} \longrightarrow \mathcal{O}_X$$

chart for the log structure on X_σ

Let ∂X_σ = complement of biggest torus orbit

$\Rightarrow \mathcal{M}_{(X_\sigma, \partial X_\sigma)} \cong P^{\log}$

On an étale open U, a chart can be thought of as a map

$$U \longrightarrow \operatorname{Spec} \mathbb{Z}[P]$$

s.t $\quad \mathcal{M}_X|_U \cong \left(P \to \mathbb{Z}[P] \to \mathcal{O}_U\right)^{\log}$

pull back of the toric boundary on $\operatorname{Spec} \mathbb{Z}[P]$

ex. (NOT fine)

$$X = \mathrm{Spec}\, k[x,y,w,t]/(xy - wt), \quad D = (t=0)$$

Theorem: (Kato's criterion)

$f : (X, \mathcal{M}_X) \longrightarrow (Y, \mathcal{M}_Y)$ fine log schemes

is <u>log smooth</u> if étale locally on $\underline{X}, \underline{Y}$

we can find a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathrm{Spec}\,\mathbb{Z}[P] \arrow[d] \\ Y \arrow[r] & \mathrm{Spec}\,\mathbb{Z}[Q] \end{tikzcd}$$

s.t ① $P \longrightarrow \mathcal{O}_X$, $Q \longrightarrow \mathcal{O}_Y$ charts

② $X \longrightarrow Y \times_{\mathrm{Spec}\,\mathbb{Z}[Q]} \mathrm{Spec}\,\mathbb{Z}[P]$ (classical) is smooth

③ $Q \longrightarrow P$ induced from $\mathrm{Spec}\,\mathbb{Z}[P] \longrightarrow \mathrm{Spec}\,\mathbb{Z}[Q]$

s.t $\mathrm{Ker}(Q^{gp} \to P^{gp})$ & torsion part of $\mathrm{Coker}(Q^{gp} \to P^{gp})$

are finite & ~~w/ order invertible on X~~

if characteristic 0

ex. $Y = (\mathrm{Spec}\, k, k^+)$

$X_\sigma = \mathrm{Spec}\, k[\sigma^\vee \cap N]$

$$\begin{tikzcd} X_\sigma \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Spec}\,\mathbb{Z}[\sigma^\vee \cap N] \arrow[d] \\ \mathrm{Spec}\, k \arrow[r] & \mathrm{Spec}\,\mathbb{Z} \end{tikzcd}$$

So X_σ is log smooth while $\underline{X_\sigma}$ not necessarily smooth.

ex. $X = \mathrm{Spec}\, k[P]$ (toric) w/ $(X, \partial X)$ $P \to k[P]$

$Y = \mathrm{Spec}\, k[\mathbb{N}] = (\mathbb{A}^1, 0)$ $\mathbb{N} \to k[\mathbb{N}]$

Suppose we have $\mathbb{N} \to P$, $1 \mapsto m$ $\rightsquigarrow$ $\mathbb{Z} \hookrightarrow \mathbb{N}$

$\therefore$ $X \to Y$ is log smooth.

In particular, the Mumford degeneration is log smooth w/ log smooth central fibre.

<u>Log tangent sheaf</u> $f : X^+ \to Y^+$

define $\Omega^1_{X^+/Y^+} = \Omega^1_{X/Y} \oplus (\mathcal{O}_X \otimes \mathcal{M}_X^{gp}) / R$

R is the $\mathcal{O}_X$-module generated by

$$d\alpha_X(m) - \alpha_X(m) \otimes m \quad \& \quad (0, 1 \otimes \pi^*(n))$$

$$d: \mathcal{O}_X \xrightarrow{d} \Omega_{X/Y} \longrightarrow \Omega_{X^+/Y^+}$$

$$d\log: M^{gp} \xrightarrow{1\otimes} \mathcal{O}_X \otimes M^{gp} \longrightarrow \Omega_X^1$$

$$d\alpha_X(m) = \alpha(m)\, d\log(m)$$

ex. $X_\sigma = \mathrm{Spec}\, k[\sigma^\vee \cap N]$

then $\Omega^1_{(X_\sigma, \partial X_\sigma)/k} = \mathcal{O}_X \otimes_{\mathbb{Z}} N$ locally free

Definition: X = variety, A <u>stable map</u> to X is a map

$$\begin{array}{c} (C, p_1, \cdots, p_n) \longrightarrow X \\ \downarrow \\ \mathrm{Spec}\, k \end{array}$$

s.t. ① C is a proper, connected, reduced, nodal alg. curve

② $p_1, \cdots, p_n$ distinct smooth points

③ If f contracts a component of genus 0, the component has 3 special points

If f contracts a component of genus 1, the component has 1 <u>special points</u>

p_i or nodal point

Definition: If $f: C \to X_\Sigma$ a stable map,
toric variety
it is <u>torically Transverse</u> if

- $f(C)$ is disjoint from every stratum of codim > 1
- no irreducible components of C maps into ∂X_Σ

Proposition: If X_Σ non-singular, then $H_2(X_\Sigma, \mathbb{Z}) \cong \ker(r)$

$$r: T_\Sigma \longrightarrow M$$
$$t_\rho \longmapsto m_\rho \text{ primitive generator}$$